

LEAFWISE COISOTROPIC INTERSECTIONS

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ABSTRACT. We establish the leafwise intersection property for closed, coisotropic submanifolds in an exact symplectic manifold satisfying natural additional assumptions.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the question of existence of leafwise intersections for coisotropic submanifolds – one of the generalizations of the well-known Lagrangian intersection property to coisotropic submanifolds. More specifically, we establish the existence of leafwise intersections for restricted contact type coisotropic submanifolds and Hamiltonian diffeomorphisms with Hofer energy below a certain natural, sharp threshold. In other words, under these conditions, we prove the existence of a leaf F of the characteristic foliation on a coisotropic submanifold M such that $\varphi(F) \cap F \neq \emptyset$ for a Hamiltonian diffeomorphism φ .

The problem of existence of coisotropic intersections was first addressed by Moser in the late 70s, [M], and since then various forms of the coisotropic intersection property have been established. For instance, the existence of ordinary intersections of M and $\varphi(M)$, or, roughly equivalently, lower bounds on the displacement energy of M have also been investigated. In this vein, positivity of the displacement energy for stable coisotropic submanifolds of \mathbb{R}^{2n} was proved by Bolle, [Bo1, Bo2]. (See Section 2.1 for definitions.) In [Gi], revitalizing the subject a decade after Bolle's work, Ginzburg extended Bolle's result to general symplectically aspherical manifolds. This was generalized to closed, rational symplectic manifolds by Kerman, [K], and, more recently, Usher, [U], proved that the displacement energy of a stable coisotropic submanifold of any closed or convex symplectic manifold is positive. Currently, the question is also being studied by Tonnelier, [T].

The main result of this paper, falling in the realm of leafwise intersections, is

Theorem 1.1. *Let (W^{2n}, ω) be an exact symplectic manifold with $c_1|_{\pi_2(W)} = 0$ which is geometrically bounded and wide. Let $M \subset W$ be a closed, coisotropic submanifold of restricted contact type. Then, for any compactly supported Hamiltonian diffeomorphism $\varphi_H: W \rightarrow W$ with $\|H\| < \mathfrak{a}(M)$, there exists a leaf F of the characteristic foliation \mathcal{F} on M such that $\varphi_H(F) \cap F \neq \emptyset$. Here,*

$$\mathfrak{a}(M) = \inf\{A(\gamma) > 0 \mid \gamma \text{ is a loop that is tangent to } \mathcal{F} \text{ and contractible in } W\},$$

where $A(\gamma)$ is the (negative) symplectic area bounded by γ .

Remark 1.2. Note that $\mathfrak{a}(M) > 0$ whenever M has restricted contact type; see [Gi]. Furthermore, we set $\mathfrak{a}(M) = \infty$ if there is no γ as in the definition of $\mathfrak{a}(M)$.

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Remark 1.3. The assumption that $c_1|_{\pi_2(W)} = 0$ is not really essential in Theorem 1.1 and included only for the sake of completeness. To be more precise, several of the results used in the proof of the theorem were originally established for symplectically aspherical manifolds. However, it is clear these results also hold for exact, wide, geometrically bounded manifolds without any additional assumptions on the first Chern class.

A symplectic manifold W is said to be wide if it is open and admits an arbitrarily large compactly supported Hamiltonian without non-trivial contractible fast periodic orbits. This notion was introduced and discussed in detail in [Gü1]. In particular, examples of wide manifolds include manifolds that are convex at infinity (e.g., \mathbb{R}^{2n} , cotangent bundles, Stein manifolds) and twisted cotangent bundles. The essence of the wideness property lies in the fact that on a wide manifold the top degree Floer homology is non-zero for any non-negative compactly supported Hamiltonian which is not identically zero. This allows one to construct an action selector for geometrically bounded and wide manifolds, a tool utilized in the proof of Theorem 1.1. (It is not known how to define action selectors for an arbitrary geometrically bounded manifold.)

As has been pointed out above, the problem of leafwise intersections was first considered in [M] where Moser established the existence of leafwise intersections whenever φ is C^1 -close to the identity and M is simply connected. The latter assumption was removed by Banyaga, [Ba]. Furthermore, Ekelund and Hofer, [EH, H], showed that for contact type hypersurfaces in \mathbb{R}^{2n} , the C^1 -smallness assumption on φ can be replaced by a much less restrictive hypothesis that the Hofer norm of φ is smaller than a certain symplectic capacity of the domain bounded by M . More recently, Ginzburg, [Gi], proved the existence of leafwise intersections for restricted contact type hypersurfaces of sub-critical Stein manifolds. The question has also been studied by Ziltener, [Z], under the additional assumption that the characteristic foliation is a fibration.

Perhaps the most relevant to this work are recent papers [AF1, AF2] by Albers and Frauenfelder. In [AF1], the authors establish the existence of leafwise intersections for hypersurfaces M of restricted contact type, provided that the ambient symplectic manifold is exact and convex at infinity and that the Hofer norm of φ does not exceed the minimal Reeb period $\mathfrak{a}(M)$. In a follow-up work, [AF2], they prove that, generically, the number of leafwise intersection points is at least the sum of \mathbb{Z}_2 -Betti numbers of M . The method used in [AF1, AF2] is the Rabinowitz Floer homology (see, e.g., [CFO]). The result of the present paper is a generalization of the theorem in [AF1] to the case where $\text{codim } M > 1$, without the multiplicity lower bound on the number of leafwise intersections. Thus, the leafwise intersection property holds not just for hypersurfaces, but is a feature of coisotropic submanifolds in general.

Remark 1.4. Comparing Theorem 1.1 with the result from [M, Ba], note that the latter holds for any coisotropic submanifold M , but the requirement on φ is very restrictive. On the other hand, in Theorem 1.1 the roles are reversed: we impose a strong condition on M while the requirement on φ has been relaxed. It is then natural ask whether Theorem 1.1 can be generalized to any coisotropic submanifold M along the lines of [M, Ba]. This, however, is not the case and leafwise intersections appear to be fragile: *There exists a hypersurface $M \subset \mathbb{R}^{2n}$ (diffeomorphic*

to S^{2n-1}) and a sequence of (autonomous) Hamiltonians $H_i: \mathbb{R}^{2n} \rightarrow \mathbb{R}$, supported within the same compact set, such that

- $\|H_i\|_{C^0} \rightarrow 0$, and
- $\varphi_{H_i}(M)$ and M have no leafwise intersections.

This result will be proved in the forthcoming paper [Gü2].

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2. PRELIMINARIES

We start this section by recalling the relevant definitions and basic results concerning coisotropic submanifolds. In Section 2.2, we set our conventions and notation. We recall the definition of the action selector for wide manifolds and state its relevant properties in Section 2.3.

2.1. Preliminaries on coisotropic submanifolds. Let (W^{2n}, ω) be a symplectic manifold and let $M \subset W$ be a closed, coisotropic submanifold of codimension k . Set $\omega_M = \omega|_M$. Then, as is well known, the distribution $\ker \omega_M$ has dimension k and is integrable. Denote by \mathcal{F} the characteristic foliation on M , i.e., the k -dimensional foliation whose leaves are tangent to the distribution $\ker \omega_M$.

Definition 2.1. The coisotropic submanifold M is said to be *stable* if there exist one-forms $\alpha_1, \dots, \alpha_k$ on M such that $\ker d\alpha_i \supset \ker \omega_M$ for all $i = 1, \dots, k$ and

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge \omega_M^{n-k} \neq 0$$

anywhere on M . We say that M has *contact type* if the forms α_i can be taken to be primitives of ω_M . Furthermore, M has *restricted contact type* if the forms α_i extend to global primitives $\bar{\alpha}_i$ of ω on W .

Stable and contact type coisotropic submanifolds were introduced by Bolle in [Bo1, Bo2] and considered in a more general setting by Ginzburg, [Gi], and afterwards also by Kerman, [K], and by Usher, [U]. (See also [D].) Referring the reader to [Gi] for a discussion of the requirements of Definition 2.1 and several illustrating examples, let us merely note that these requirements are natural but quite restrictive. For example, a stable Lagrangian submanifold is necessarily a torus. In this paper, we will be mainly concerned with coisotropic submanifolds of restricted contact type. This condition is a generalization of its namesake for hypersurfaces, as is the case for contact type or stability conditions.

Assume that M is stable. The normal bundle to M in W is trivial since it is isomorphic to $T^*\mathcal{F}$ and, thus it can be identified with $M \times \mathbb{R}^k$. From now on we identify a small neighborhood of M in W with a neighborhood of M in $T^*\mathcal{F} = M \times \mathbb{R}^k$ and use the same symbols ω_M and α_i for differential forms on M and for their pullbacks to $M \times \mathbb{R}^k$. (Thus, we will be suppressing the pullback notation π^* , where $\pi: M \times \mathbb{R}^k \rightarrow M$, unless its presence is absolutely necessary.) Using the Weinstein symplectic neighborhood theorem, we then have

Proposition 2.2 ([Bo1, Bo2]). *Let M be a closed, stable coisotropic submanifold of (W^{2n}, ω) with $\text{codim } M = k$. Then, for a sufficiently small $r > 0$, there exists a neighborhood of M in W which is symplectomorphic to $U_r = \{(q, p) \in M \times \mathbb{R}^k \mid |p| < r\}$ equipped with the symplectic form $\omega = \omega_M + \sum_{j=1}^k d(p_j \alpha_j)$. Here (p_1, \dots, p_k) are the coordinates on \mathbb{R}^k and $|p|$ is the Euclidean norm of p .*

As an immediate consequence of Proposition 2.2, we obtain a family of coisotropic submanifolds $M_p = M \times \{p\}$, for $p \in B_r^k$, of W which foliate a neighborhood of M in W , where B_r^k is the ball of radius r , centered at the origin in \mathbb{R}^k . Moreover, a leaf of the characteristic foliation on M_p projects onto a leaf of the characteristic foliation on M .

Furthermore, we have

Proposition 2.3 ([Bo1, Bo2, Gi]). *Let M be a stable coisotropic submanifold.*

- (i) *The leafwise metric $(\alpha_1)^2 + \cdots + (\alpha_k)^2$ on \mathcal{F} is leafwise flat.*
- (ii) *The Hamiltonian flow of $\rho = (p_1^2 + \cdots + p_k^2)/2 = |p|^2/2$ is the leafwise geodesic flow of this metric.*

Remark 2.4. Using this property, it is not hard to show that, whenever M has restricted contact type, for every leaf F of \mathcal{F} , the kernel of the map $H_1(F; \mathbb{R}) \rightarrow H_1(W; \mathbb{R})$ is either trivial or one dimensional. For instance, if $H_1(W; \mathbb{R}) = 0$, every leaf F is diffeomorphic to either \mathbb{R}^k or $S^1 \times \mathbb{R}^{k-1}$.

2.2. Conventions and notation. In this section we set our conventions and notation.

Let (W^{2n}, ω) be a symplectically aspherical manifold, i.e., $\omega|_{\pi_2(W)} = c_1|_{\pi_2(W)} = 0$. Denote by ΛW the space of smooth contractible loops $\gamma: S^1 \rightarrow W$ and consider a time-dependent Hamiltonian $H: S^1 \times W \rightarrow \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. Setting $H_t = H(t, \cdot)$ for $t \in S^1$, we define the action functional $A_H: \Lambda W \rightarrow \mathbb{R}$ by

$$A_H(\gamma) = A(\gamma) + \int_{S^1} H_t(\gamma(t)) dt,$$

where $A(\gamma)$ is the negative symplectic area bounded by γ , i.e.,

$$A(\gamma) = - \int_{\bar{\gamma}} \omega,$$

where $\bar{\gamma}: D^2 \rightarrow W$ is such that $\bar{\gamma}|_{S^1} = \gamma$.

The least action principle asserts that the critical points of A_H are exactly contractible one-periodic orbits of the time-dependent Hamiltonian flow φ_H^t of H , where the Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. We denote the collection of such orbits by \mathcal{P}_H . The action spectrum $\mathcal{S}(H)$ of H is the set of critical values of A_H . In other words, $\mathcal{S}(H) = \{A_H(\gamma) \mid \gamma \in \mathcal{P}_H\}$. This is a zero measure set; see, e.g., [HZ].

In what follows we will always assume that H is compactly supported and set $\text{supp } H = \bigcup_{t \in S^1} \text{supp } H_t$. In this case, $\mathcal{S}(H)$ is compact and hence nowhere dense.

Let $J = J_t$ be a time-dependent almost complex structure on W . A Floer anti-gradient trajectory u is a map $u: \mathbb{R} \times S^1 \rightarrow W$ satisfying the equation

$$\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = -\nabla H_t(u).$$

Here, the gradient is taken with respect to the time-dependent Riemannian metric $\omega(\cdot, J_t \cdot)$. Denote by $u(s)$ the curve $u(s, \cdot) \in \Lambda W$.

The energy of u is defined as

$$E(u) = \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|_{L^2(S^1)}^2 ds = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial t} - J \nabla H(u) \right\|^2 dt ds.$$

We say that u is asymptotic to $x^\pm \in \mathcal{P}_H$ as $s \rightarrow \pm\infty$ or connecting x^- and x^+ if $\lim_{s \rightarrow \pm\infty} u(s) = x^\pm$ in ΛW . More generally, u is said to be partially asymptotic to $x^\pm \in \mathcal{P}_H$ at $\pm\infty$ if $u(s_n^\pm) \rightarrow x^\pm$ for some sequences $s_n^\pm \rightarrow \pm\infty$. In this case

$$A_H(x^-) - A_H(x^+) = E(u).$$

In this paper the manifold W is assumed to be exact and, hence, open. In this case, in order for the Floer homology to be defined, we assume that W is geometrically bounded. This assumption gives us sufficient control of the geometry of W at infinity which is necessary in the case of open manifolds. Examples of such manifolds include symplectic manifolds that are convex at infinity (e.g., \mathbb{R}^{2n} , cotangent bundles) as well as twisted cotangent bundles. (See, e.g., [AL, CGK] for the precise definition and a discussion of geometrically bounded manifolds.) Under the hypotheses that W is symplectically aspherical and geometrically bounded, the compactness theorem for Floer's connecting trajectories holds and the filtered \mathbb{Z} -graded Floer homology of a compactly supported Hamiltonian on W is defined; see, e.g., [CGK, GG].

2.3. Action selector for wide manifolds. In this section we briefly recall the definition and relevant properties of the action selector defined in [Gü1] for wide and geometrically bounded manifolds. We refer the reader to [Gü1, Gi] for more details. Here we only note that action selectors were constructed in [S] for closed manifolds and in [HZ] and [V] for \mathbb{R}^{2n} and cotangent bundles, respectively. The approach of [S] has been extended to manifolds convex at infinity in [FS].

The definition. Let $H: S^1 \times W \rightarrow \mathbb{R}$ be a compactly supported, non-negative Hamiltonian which is not identically zero. As was proved in [Gü1], on a wide manifold, the top degree Floer homology of H for the action interval $(0, \infty)$ is non-zero and it carries a canonically defined homology class. (Note that the homology group itself depends on H .) Call this class $[\max H]$ and define

$$\sigma(\varphi_H) = \inf\{a > 0 \mid j_H^a([\max H]) = 0\} \in \mathcal{S}(H),$$

where

$$j_H^a: \mathrm{HF}^{(0, \infty)}(H) \rightarrow \mathrm{HF}^{(a, \infty)}(H)$$

is the quotient map. (This definition coincides with the one from [FS] whenever W is convex.)

Properties of the action selector. Focusing on the ones that are relevant for what follows, recall that the action selector σ , defined as above, has the following properties for non-negative Hamiltonians:

- (S1) σ is monotone, i.e., $\sigma(\varphi_K) \leq \sigma(\varphi_H)$, whenever $0 \leq K \leq H$ point-wise;
- (S2) $0 \leq \sigma(\varphi_H) \leq E^+(H)$ for any $H \geq 0$, where

$$E^+(H) = \int_{S^1} \max_W H_t dt;$$

- (S3) $\sigma(\varphi_H) > 0$, provided that $H \geq 0$ is not identically zero;
- (S4) $\sigma(\varphi_H)$ is continuous in H in the C^0 -topology.

We refer the reader to [Gü1] for the proofs of these properties.

3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. Throughout the proof, as in Section 2.1, a neighborhood of M in W is identified with a neighborhood of M in $M \times \mathbb{R}^k$ equipped with the symplectic form $\omega = \omega_M + \sum_{j=1}^k d(p_j \alpha_j)$. Using this identification, we denote by U_R , with $R > 0$ sufficiently small, the neighborhood of M in W corresponding to $M \times B_R^k$. (Thus, $U_R = \{\rho < R^2/2\}$.)

Proof of Theorem 1.1. First note that, without loss of generality, we may assume that $\text{Fix}(\varphi_H) \cap M = \emptyset$, for otherwise the assertion is obvious. Then φ_H has no fixed points near M , say in a tubular neighborhood $U = M \times B_R^k$ of M in W , where $R > 0$ is sufficiently small. Let $U_r \subset U$ be a smaller tubular neighborhood of M for some $r < R$. We may also require that $H \geq 0$ and $\|H\| = E^+(H)$. (This can be achieved by replacing H by $f \cdot (H - \min H)$, where f is a cut-off function equal to one near $\text{supp}(H) \cup \bar{U}$.)

Let $K \geq 0$ be a non-negative function on $[0, r]$ such that

- $K(0) = \max K > 0$ and $K \equiv 0$ near r ;
- K is strictly decreasing until it becomes zero;
- all odd-order derivatives of K vanish at 0, and $K''(0) < 0$ is close to zero.

Abusing notation, we also denote by K the function on W , equal to $K(|p|)$ on U_r , where $|p| = \sqrt{2\rho}$, and extended to be identically zero outside U_r . Note that K on W has only two critical values: $\max K$ and 0. Observe also that the Hamiltonian flow of K on U_r is just a reparametrization of the leafwise geodesic flow on M and the flow is the identity map outside U_r .

Recall that $\mathfrak{a}(M) = \inf\{A(\gamma) > 0 \mid \gamma \text{ is tangent to } \mathcal{F} \text{ and contractible in } W\} > 0$, where $A(\gamma)$ denotes the negative symplectic area bounded by the orbit γ , and set $\mathfrak{a}(M) = \infty$ if there is no such γ .

The first ingredient in the proof of Theorem 1.1 is

Lemma 3.1. *In the above setting, there exists a constant $C > 0$, depending on U_r but not on K , such that whenever $\max K \geq C$, we have $\sigma(\varphi_K) \geq C$. Moreover, C can be taken to be of the form $C = (1 - O(r))\mathfrak{a}(M)$. If $\mathfrak{a}(M) = \infty$, the constant C can be taken to be arbitrarily large.*

Remark 3.2. A version of this lemma also holds when M is just stable although in this case we cannot guarantee that C has the desired form. However, one can then take C to be independent of r (c.f. Theorem 2.7 of [Gi]).

Proof of Lemma 3.1. Set $T_x^\perp \mathcal{F} = \cap \ker \alpha_j \subset T_{(x,p)} M_p$. Then,

$$T_{(x,p)} U = T_x^\perp \mathcal{F} \times (T_x \mathcal{F} \times T_p B) \quad (3.1)$$

is a decomposition of the tangent space at a point $(x, p) \in U$ into a direct sum of two symplectic subspaces. Indeed, observe that

$$\omega_{(x,p)} = \left(1 + \sum_{j=1}^k p_j\right) \omega_x|_{T_x^\perp \mathcal{F}} + \sum_{j=1}^k dp_j \wedge \alpha_j,$$

where the first symplectic form vanishes on $T_x \mathcal{F} \times T_p B$ and the second one vanishes on $T_x^\perp \mathcal{F}$.

Let us now introduce an almost complex structure $J = J(x, p)$ on W , compatible with ω and (3.1). To this end, observe that the non-degeneracy condition on α_i 's

implies that the forms are linearly independent and leafwise closed. Let X_1, \dots, X_k be the basis in $T_x \mathcal{F}$ dual to $\alpha_1, \dots, \alpha_k$, i.e., $\alpha_i(X_j) = \delta_{ij}$. Clearly, $\partial_{p_1}, \dots, \partial_{p_k}$ form a basis in $T_p B$. Define J such that $T_x^\perp \mathcal{F}$ and $T_x \mathcal{F} \times T_p B$ are complex subspaces in $T_{(x,p)} U$; furthermore, $J|_{T_x^\perp \mathcal{F}}$ is compatible with $\omega_x|_{T_x^\perp \mathcal{F}}$; and on $T_x \mathcal{F} \times T_p B$, we have $J(X_i) = \partial_{p_i}$. (Outside U_R , we take J to be an arbitrary almost complex structure compatible with ω .) The pair (ω, J) gives rise to a metric on M compatible with ω and such that (3.1) is an orthogonal decomposition.

With respect to this metric, $\|\partial_{p_i}\| = 1$ and $\|X_i\| = 1$. Moreover, $\|\omega\| := \sup_{X,Y} \omega(X,Y) = 1$, where the supremum is taken over all tangent vectors X, Y with norm one. Note also that the Hamiltonian vector field of $\rho = (p_1^2 + \dots + p_k^2)/2$ is $X_\rho = \sum_{j=1}^k p_j X_j$ and, thus, $J(X_\rho) = \sum_{j=1}^k p_j \partial_{p_j}$.

We may assume without loss of generality that the forms α_i extend to primitives $\bar{\alpha}_i$ of ω on W such that

$$\bar{\alpha}_i = \alpha_i + \sum_{j=1}^k p_j \alpha_j \text{ on } U_R.$$

In the spirit of [Bo1, Bo2, Gi], our next goal is to define smooth one-forms β_i on W for $i = 1, \dots, k$ such that β_i agrees with α_i on U_r and with $\bar{\alpha}_i$ outside U_R and that

$$\|d\beta_i\|_{C^0} \leq 1 + O(r).$$

To this end, let $g = g_r(y)$ be a family of smooth, non-negative, monotone increasing functions defined on $[0, \infty)$ such that $g \equiv 0$ on $[0, r]$ and $g \equiv 1$ on $[R, \infty)$, and that $0 \leq g'(y)y + g(y) \leq 1 + O(r)$. It is not hard to write an explicit formula for such functions g . Abusing notation, denote also by g the function $g(|p|)$ defined on W . Finally, define the one-forms β_i as

$$\beta_i = \begin{cases} \alpha_i + g(|p|) \sum_{j=1}^k p_j \alpha_j & \text{on } U_R \\ \bar{\alpha}_i & \text{outside } U_R \end{cases}$$

Now a straightforward, but tedious, calculation shows that $\|d\beta_i\|_{C^0} \leq 1 + O(r)$. It is particularly easy to see that this is the case when M is a hypersurface. Namely, then $k = 1$ and $d\beta = (1 + f)d\alpha + df \wedge \alpha$, where $f(y) = yg(y)$. Notice that we only need to prove the desired estimate on U_R . Then, $d\alpha = \pi^* \omega_M$ and the first form vanishes on $T_x \mathcal{F} \times T_p B$, while the second form vanishes on $T_x^\perp \mathcal{F} = \ker \alpha$. Thus, it suffices to prove the estimate on these two symplectic subspaces separately. To this end, observe that on U_R we have

$$\|d\beta|_{T_x^\perp \mathcal{F}}\| = (1 + f)/(1 + p) = 1 + O(r),$$

as is easy to see. On the other hand, $T_x \mathcal{F} \times T_p B$ is spanned by $\{X, \partial_p\}$ and

$$\|d\beta|_{T_x \mathcal{F} \times T_p B}\| = |d\beta(X, \partial_p)|.$$

By the definition of g , and, hence, of f , we have $d\beta(X, \partial_p) = f' \leq 1 + O(r)$. The calculation in the general case is more involved but follows the same track as the one for hypersurfaces.

A feature of the form β_i , important in what follows, is that

$$i_{X_K} d\beta_i = 0. \tag{3.2}$$

To see this, observe first that (3.2) trivially holds outside U_r , for $\text{supp } K \subset U_r$. Moreover, $d\beta_i = d\pi^*\alpha_i = \pi^*\omega_M$ on U_r and we then have

$$i_{X_K} d\beta_i = i_{X_K} \pi^* d\alpha_i = K'(\rho) i_{\pi_* X_\rho} d\alpha_i = 0.$$

The last equality follows from the fact that $\pi_* X_\rho$ is tangent to \mathcal{F} by Proposition 2.3 and that $T\mathcal{F} \subset \ker d\alpha_i$ since M is stable.

Let $\epsilon > 0$ be small enough so that $f = \epsilon K$ is C^2 -small and consider a linear homotopy from K to f , running through functions of ρ . Then, there exist an orbit γ of K and a homotopy trajectory u partially asymptotic to γ at $-\infty$ and to a critical point of f (and, hence, of K) on M at ∞ such that

- (i) $A_K(\gamma) \leq \sigma(\varphi_K)$, and
- (ii) $E(u) \leq A_K(\gamma) - \epsilon \max K \leq \sigma(\varphi_K) - \epsilon \max K$.

This fact is essentially a particular case of Proposition 5.4 in [Gi] with the displacability requirement being inessential for us and it can be proved exactly in the same way.

We claim that

$$E(u) \geq \|d\beta_i\|^{-1} \left| \int_{\pi(\gamma)} \alpha_i \right|. \quad (3.3)$$

To prove (3.3), fix $s_n^\pm \rightarrow \pm\infty$ such that $u(s_n^+)$ converges to a point of M and $u(s_n^-)$ converges to γ in $C^\infty(S^1, W)$. Then, using (3.2), we have

$$\begin{aligned} E(u) &= \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial s} \right\| \cdot \left\| \frac{\partial u}{\partial t} - X_K \right\| dt ds \\ &\geq \|d\beta_i\|^{-1} \int_{-\infty}^{\infty} \int_{S^1} \left| d\beta_i \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_K \right) \right| dt ds \\ &\geq \|d\beta_i\|^{-1} \liminf_{n \rightarrow \infty} \left| \int_{s_n^-}^{s_n^+} \int_{S^1} d\beta_i \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_K \right) dt ds \right| \\ &= \|d\beta_i\|^{-1} \liminf_{n \rightarrow \infty} \left| \int_{s_n^-}^{s_n^+} \int_{S^1} d\beta_i \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) dt ds \right|. \end{aligned}$$

By Stokes' formula,

$$\left| \int_{s_n^-}^{s_n^+} \int_{S^1} d\beta_i \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) dt ds \right| = \left| \int_{u(s_n^+)} \beta_i - \int_{u(s_n^-)} \beta_i \right| \longrightarrow \left| \int_{\gamma} \beta_i \right|$$

as $n \rightarrow \infty$. Furthermore, recall that γ is contained in U_r and $\beta_i|_{U_r} = \pi^*\alpha_i$. Thus,

$$\left| \int_{\gamma} \beta_i \right| = \left| \int_{\gamma} \pi^* \alpha_i \right| = \left| \int_{\pi(\gamma)} \alpha_i \right|,$$

which completes the proof of (3.3).

Set $C = \|d\beta_i\|^{-1} \mathfrak{a}(M)$, assuming that $\mathfrak{a}(M) < \infty$. It is clear that then $C = (1 - O(r)) \mathfrak{a}(M) > 0$. Assume now that $\max K \geq C$ and recall that $\sigma(\varphi_K) \leq \max K$. If $\sigma(\varphi_K) = \max K$, the proof is finished. Let us focus on the case when $\sigma(\varphi_K) < \max K$. Then, $\gamma \subset U_r$ is necessarily nontrivial. Indeed, trivial orbits of K occur where K is constant, i.e. $K = 0$ or $K = \max K$. Since (i) implies that $A_K(\gamma) \leq \sigma(\varphi_K) < \max K$, we must have $K = 0$ whenever γ is constant and, hence, $A_K(\gamma) = 0$. Consequently, we infer from (ii) that $E(u) < 0$, which is clearly a

contradiction. Hence, γ is nontrivial and, as an immediate consequence, we have $\left| \int_{\pi(\gamma)} \alpha_i \right| > 0$. Then, by definition of $\mathfrak{a}(M)$,

$$\sigma(\varphi_K) \geq E(u) \geq \|d\beta_i\|^{-1} \left| \int_{\pi(\gamma)} \alpha_i \right| \geq \|d\beta_i\|^{-1} \mathfrak{a}(M) = C.$$

Finally, note that if $\mathfrak{a}(M) = \infty$, we must necessarily have $\sigma(\varphi_K) = \max K$ and, then, C can be taken to be arbitrarily large. \square

Returning to the proof of Theorem 1.1, let $C > 0$ be a constant as in Lemma 3.1 and assume from now on that the function K has the additional property that $\max K = 2C > C$. Thus, $\sigma(\varphi_K) \geq C$.

For the sake of convenience, let us reparametrize H and K , as functions of s and t , so that $H_t \equiv 0$ on $[0, 1/2]$ and $K_t \equiv 0$ on $[1/2, 1]$. (Note that the time-one maps, the Hofer norms, the action spectra, and the action selectors remain the same.) Then, the flow given by (a smooth reparametrization of) the concatenation of paths φ_K^t and $\varphi_H^t \varphi_K$, where $t \in [0, 1]$, is homotopic with fixed end points to $\varphi_H^t \varphi_K^t$, and is generated by the Hamiltonian $H_t + K_t$.

Consider the family of diffeomorphisms $\psi_s^t = \varphi_H^t \varphi_{sK}^t$ for $s \in [0, 1]$, starting at $\psi_0^t = \varphi_H^t$ and ending at $\psi_1^t = \varphi_H^t \varphi_K^t$. Note that, for a fixed s , we have $\psi_s^0 = id$ and $\psi_s^1 = \psi_s = \varphi_H \varphi_{sK}$, and, thus each ψ_s^t , as an element of $\widetilde{\text{Ham}}$, is generated by the Hamiltonian $H_t + sK_t =: G_t^s$. Examining the fixed points of ψ_s , as is easy to see using the assumption that $\text{Fix}(\varphi_H) \cap U_r = \emptyset$, we have the disjoint union decomposition

$$\text{Fix}(\psi_s) = \text{Fix}(\varphi_H) \sqcup Z_s, \text{ where } Z_s = \{x \in U_r \mid \varphi_H^{-1}(x) = \varphi_{sK}(x)\}.$$

Furthermore, $\sigma(\psi_s)$, as a function of s , is not constant. Indeed, first note that, by our assumption on H ,

$$\sigma(\psi_0) = \sigma(\varphi_H) \leq \|H\| < C.$$

Since $H \geq 0$ and σ is monotone, we also have the inequality

$$\sigma(\psi_1) = \sigma(\varphi_H \varphi_K) \geq \sigma(\varphi_K).$$

Finally, using Lemma 3.1 along with these two inequalities, we obtain

$$\sigma(\psi_0) < C \leq \sigma(\varphi_K) \leq \sigma(\psi_1).$$

This shows that $\sigma(\psi_s)$ is not constant. On the other hand, $\sigma(\psi_s)$ is a continuous function of s and the set $\text{Fix}(\varphi_H)$ is nowhere dense. Thus, $Z_{s_0} \neq \emptyset$ for some $s_0 \in (0, 1]$ and $\sigma(\psi_{s_0})$ is the action value of ψ_{s_0} on a fixed point $x \in Z_{s_0}$. (Notice that $Z_0 = \text{Fix}(\varphi_H) \cap U_r = \emptyset$.)

Let $\gamma(t)$ be the one-periodic orbit of the time-dependent flow $\psi_{s_0}^t = \varphi_H^t \varphi_{s_0 K}^t$, passing through $x = \gamma(0)$. This flow is generated by $G_t^{s_0} = H_t + s_0 K_t$ and, due to the above parametrizations of H and K , the orbit γ is comprised of two parts: $\gamma_K(t) = \varphi_{s_0 K}^t(x)$ – a trajectory of $s_0 K$ beginning at x and ending at $y = \varphi_{s_0 K}(x)$ – and $\gamma_H(t) = \varphi_H^t(y)$ – a trajectory of H beginning at y and ending at x . Note that x and y lie on the same leaf of the characteristic foliation on some M_p and $\varphi_H(y) = x$, where $p \in B_r^k$ such that $0 < |p| < r$.

Furthermore,

$$\sigma(\psi_{s_0}) = A_{G^{s_0}}(\gamma) = - \int_{\gamma_K} \lambda + \int_0^{1/2} s_0 K_t(\gamma_K(t)) dt + A_H(\gamma_H),$$

where λ is a global primitive of ω . Our next goal is to show that

$$\left| \int_{\gamma_K} \lambda \right| = \left| -\sigma(\psi_{s_0}) + \int_0^{1/2} s_0 K_t(\gamma_K(t)) dt + A_H(\gamma_H) \right| \quad (3.4)$$

is bounded by a constant independent of K , which will in turn imply that the time required to move x to y is (uniformly) bounded. To see this, first recall that

$$0 < \sigma(\psi_{s_0}) \leq \|H_t + s_0 K_t\| \leq \|H\| + \|K\| \leq C + 2C = 3C.$$

The second term in (3.4) is bounded from below by zero since $K \geq 0$ and from above by $\max K/2 = C$. Finally, $|A_H(\gamma_H)|$ is bounded. Indeed, consider the function $h(z) = A_H(\varphi_H^t(z))$, where $t \in [0, 1]$, defined using the primitive λ . (For example, $h(y) = A_H(\gamma_H)$.) This function is compactly supported and independent of K . Letting $C' = \max\{|\max h|, |\min h|\}$, a constant independent of K , we see that $|\int_{\gamma_K} \lambda| \leq 4C + C'$.

Let $X = X_\rho/|p|$ and note that $\alpha_i(X) = p_i/|p|$, where $|p| = \sqrt{2\rho}$. Consider the primitives $\lambda = \bar{\alpha}_i$ of ω for $i = 1, \dots, k$. As was shown above,

$$\left| \int_{\gamma_K} \lambda \right| = \left| \int_{\gamma_K} \bar{\alpha}_i \right| \leq C_i,$$

where $C_i > 0$ is a constant independent of K . On the other hand, on U_R we have

$$\bar{\alpha}_i(X) = \alpha_i(X) + \sum_{j=1}^k p_j \alpha_j(X) = \frac{p_i}{|p|} + |p|.$$

As a result,

$$\int_{\gamma_K} \bar{\alpha}_i = \int_0^T \bar{\alpha}_i(X) dt = T \left(\frac{p_i}{|p|} + |p| \right),$$

where T is the time required for the flow of X to move x to y . For $\bar{C} = \sum_{i=1}^k C_i$, we then have

$$\begin{aligned} \bar{C} &\geq \sum_{i=1}^k \left| \int_{\gamma_K} \bar{\alpha}_i \right| = \sum_{i=1}^k T \left| \frac{p_i}{|p|} + |p| \right| \\ &\geq T \left(\sum_{i=1}^k \frac{|p_i|}{|p|} - k|p| \right) \\ &\geq T(c - k|p|), \end{aligned}$$

where $c = \min_{|p|=1} \sum_{i=1}^k |p_i| > 0$. Note that $|p| < r$. Thus, when $r < c/2k$, we have $\bar{C} \geq Tc/2$. Hence $T \leq 2\bar{C}/c$, where the right-hand side is independent of K .

As the final step of the proof, consider a sequence $r_i \rightarrow 0$ and a sequence of Hamiltonians K_i such as K , supported on the (smaller and smaller) neighborhood U_{r_i} of M . For each K_i , we have a pair of points x_i and y_i lying on the same leaf of the characteristic foliation on some $M_{p(i)}$ with $|p(i)| \rightarrow 0$ and such that $\varphi_H(y_i) = x_i$. Furthermore, the time T_i required to move x_i to y_i is bounded from above by a constant independent of K_i and r_i . Applying the Arzela–Ascoli theorem and passing if necessary to a subsequence, we obtain points $x = \lim x_i$ and $y = \lim y_i$ on M lying on the same leaf of the characteristic foliation on M and such that $\varphi_H(y) = x$. This completes the proof of Theorem 1.1. \square

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